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Learning from World Trade Center and Building Demolitions

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Mechanics of Progressive Collapse:
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Abstract: Progressive collapse is a failure mode of great concern for tall buildings, and is also typical of building demolitions. The most infamous paradigm is the collapse of World Trade Center towers. After reviewing the mechanics of their collapse, the motion during the crushing of one floor (or group of floors) and its energetics are analyzed, and a one-dimensional continuum model of progressive collapse is developed. Rather than using classical homogenization, it is found more effective to characterize the continuum by en energetically equivalent snap-through. The collapse, in which two phases—crush-down followed by crush-up—must be distinguished, is described in each phase by a nonlinear second-order differential equation for the propagation of the crushing front of a compacted block of accreting mass. Expressions for consistent energy potentials are formulated and an exact analytical solution of a special case is given. It is shown that progressive collapse will be triggered if the total (internal) energy loss during the crushing of one story (equal to the energy dissipated by the complete crushing and compaction of one story, minus the loss of gravity potential during the crushing of that story) exceeds the kinetic energy impacted to that story. Regardless of the load capacity of the columns, there is no way to deny the inevitability of progressive collapse driven by gravity *alone* if this criterion is satisfied (for the World Trade Center it is, with an order-of-magnitude margin). The parameters are the compaction ratio of a crushed story, the fracture of mass ejected outside the tower perimeter, and the energy dissipation per unit height. The last is the most important, yet the hardest to predict theoretically. Using inverse analysis, one could identify these parameters from a precise record of the motion of floors of a collapsing building. Due to a shroud of dust and smoke, the videos of WTC are useless here. It is proposed to obtain such records by monitoring the precise time history of displacements in different modes of building demolitions. The monitoring could be accomplished by real-time telemetry from sacrificial accelerometers, or by high-speed optical camera. The resulting information on energy absorption capability would be valuable for the rating of various structural systems and for inferring their collapse mode under extreme fire, internal explosion, external blast, impact or other kinds of terrorist attack, as well as earthquake and foundation movements.

Introduction

The destruction of the World Trade Center (WTC) on 9/11/01 was not only the biggest mass murder in the U.S. history but also a big surprise for the structural engineering profession, perhaps the biggest since the collapse of Tacoma Bridge in 1940. No experienced structural engineer watching the attack expected the WTC towers to collapse. No skyscraper has ever before collapsed due to fire. The fact that the WTC towers did, beckons deep examination.

In this paper,[3] attention will be focused on the progressive collapse, triggered in WTC by fire and previously experienced in many tall buildings as a result of earthquake or explosions

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(including terrorist attack). A simplified one-dimensional analytical solution of the collapse front propagation will be presented. It will be shown how this solution can be used to determine the energy absorption capability of individual stories if the motion history is precisely recorded. Because of the shroud of dust and smoke, these histories cannot be identified from the videos of the collapsing WTC towers, and so nothing can be learned in this regard from that collapse. However, monitoring of tall building demolitions, which represent a kind of progressive collapse, could provide such histories, and using different demolition modes can provide further valuable data. Development of a simple theory amenable to inverse analysis of these histories is the key. It would permit extracting valuable information on the energy absorption capability of various types of structural systems, and is, therefore, the main objective of this paper.

Many disasters other than WTC attest to the danger of progressive collapse—e.g., the collapse of Ronan Point apartments in U.K., 1968 (Levy and Salvadori 1992), where kitchen gas explosion on the 18th floor sent a 25-story stack of rooms to the ground; the bombing of Murrah Federal Building in Oklahoma City, 1995, where the air blast pressure sufficed to take out only a few lower floors, while the upper floors failed by progressive collapse; the 2000 Commonwealth Ave. tower in Boston, 1971, triggered by punching of insufficiently hardened slab; the New World Hotel in Singapore; many buildings in Armenian earthquake, Turkish, Mexico City and other earthquake; etc. A number of ancient towers failed in this way, too—e.g., the Civic Center of Pavia in 1989 (Binda et al. 1992); the cathedral in Goch, Germany; the Campanile in Venice in 1902; etc. (Heinle and Leonhardt 1989), where the trigger was centuries-long stress redistribution due to drying shrinkage and creep (Ferretti and Bažant 2006).

Review of Causes of WTC Collapse

Although the structural damage inflicted by aircraft was severe, it was only local. Without stripping of much of the steel insulation during impact, the subsequent fire could not have led to overall collapse (Bažant and Zhou 2002, NIST 2005). As generally accepted in structural engineering and structural mechanics community (though not among some laymen seeking to unveil a conspiracy), the failure scenario was as follows:

1. About 60% of the 60 columns of the impacted face of framed-tube (and about 13% of the total of 287 columns) were severed, and many more were significantly deflected. This caused stress redistribution, which significantly increased the load of some columns, attaining or nearing the load capacity for some of them.

2. Significant amount of fire insulation was stripped during aircraft impact by flying debris (without that, the towers would likely have survived). In consequence, many structural steel members heated up to 600°C (NIST 2005) [the structural steel used loses about 20% of its yield strength already at 300°C, and about 85% at 600°C, NIST 2005; and exhibits significant visco-plasticity, or creep, above 450° (e.g. Cottrell 1964, p. 299), especially in the columns overloaded by load redistribution; the press reports right after 9/11, indicating temperature in excess of 800°C, turned out to be groundless, but Bažant and and Zhou's analysis did not depend on that].

3. Differential thermal expansion, combined with heat-induced viscoplastic deformation, caused the floor trusses to sag. The sagging trusses pulled many perimeter columns inward (by about 1 m, NIST 2005). The bowing of these columns served as a huge imperfection inducing multi-story out-of-plane buckling of framed tube wall. The lateral

deflections of some columns due to aircraft impact and differential thermal expansion also decreased buckling strength.

4. The combination of six effects—a) overload of some columns due to initial stress redistribution, b) great lowering of yield limit and creep, c)lateral deflections of many columns due to thermal strains and sagging floor trusses, d) weakened lateral support due to reduced in-plane stiffness of sagging floors, e) multi-story buckling of some columns (for which the critical load is an order of magnitude less than it is for one-story buckling), and f) local plastic buckling of heated column webs—finally led to buckling of columns (Fig. 1b). As a result, the upper part of tower fell, with little resistance, through at least one floor height, impacting the lower part of tower. This triggered progressive collapse because the kinetic energy of the falling upper part exceeded (by an order of magnitude) the energy that could be absorbed by limited plastic deformations and fracturing in the lower part of tower.

In broad terms, this scenario was proposed by Bažant (2001), and Bažant and Zhou (2002) on the basis of simplified analysis relying on energy considerations. Up to the moment of collapse trigger, the foregoing scenario was identified by meticulous, exhaustive and very realistic computer simulations of unprecedented detail, conducted S. Shyam Sunder's team at the National Institute of Standards and Technology (NIST). The progressive collapse was not simulated at NIST because its inevitability, once triggered by column buckling, had already been proven by Bažant and Zhou's (2002) comparison of kinetic energy to energy absorption capability.

Before disappearing from view, the upper part of the South tower was seen to tilt significantly (and of the North tower mildly). Some wondered why the tilting (Fig. 1d) did not continue, so that the upper part would pivot about its base like a falling tree (see Fig. 4 of Bažant and Zhou 2002). However, such toppling to the side was impossible because the horizontal reaction to the rate of angular momentum of the upper part would have exceeded the elasto-plastic shear resistance of the story at least 10.3× (Bažant and Zhou 2002).

The kinetic energy of the top part of tower impacting the floor below was found to be about 8.4× larger than the plastic energy absorption capability of the underlying story, and considerably higher than that if fracturing were taken into account (Bažant and Zhou 2002). This fact, along with the fact that, during the progressive collapse of underlying floors (Figs. 1d and 2) the kinetic energy rapidly increases (roughly in proportion to the square of the number of stories traversed), sufficed to Bažant and Zhou (2002) to conclude that the tower was doomed once the top part of tower has dropped through the height of one story (or even 0.5 m). It was also observed that this conclusion made any calculations of the dynamics of progressive collapse after the first single-story drop of upper part superfluous. The relative smallness of energy absorption capability compared to the kinetic energy also sufficed to explain, without any further calculations, why the collapse duration could not have been much longer (say, twice as long or more) than the duration of a free fall from the tower top.

Therefore, no further analysis has been necessary to prove that the WTC towers had to fall the way they did, due to gravity alone. However, a theory describing the progressive collapse dynamics beyond the initial trigger, with WTC as a paradigm, could nevertheless be very useful for other purposes, especially for learning from demolitions. Formulating this theory is the main objective of what follows.

Motion of Crushing Columns of One Story and Energy Dissipation

When the upper floor crashes into the lower one, with a layer of rubble between them, the initial height h of the story is reduced to λh, with λ denoting the compaction ratio (in finite-strain theory, λ is called the stretch). After that, the load can increase without bounds. In a one-dimensional model pursued here, one may use the estimate:

$$\lambda = (1 - \kappa_{out})V_1/V_0 \tag{1}$$

where V_0 = initial volume of the tower, $V_1 \approx$ volume of the rubble on the ground into which the whole tower mass has been compacted, and κ_{out} = correction representing mainly the fraction of the rubble that has been ejected during collapse outside the perimeter of the tower and thus does not resist compaction. The rubble that has not been ejected during collapse but was pushed outside the tower perimeter only after landing on the heap on the ground should not be counted in κ_{out}. The volume of the rubble found outside the footprint of the tower, which can be measured by surveying the rubble heap on the ground after the collapse, is an upper bound on V_1, but probably much too high a bound for serving as an estimate.

Let u denote the vertical displacement of the top floor relative to the floor below (Fig. 3, 4), and $F(u)$ the corresponding vertical load that all the columns of the floor transmit. To analyze progressive collapse, the complete load-displacement diagram $F(u)$ must be known (Fig. 3, 4 top left). It begins by elastic shortening and, after the peak load, F steeply declines with u due to plastic buckling, combined with fracturing (for columns heated above approximately 450°C, the buckling is viscoplastic). For single column buckling, the inelastic deformation localizes into three plastic (or softening) hinges (see Figs. 2b,c and 5b in Bažant and Zhou 2002). For multi-story buckling, the load-deflection diagram has a similar shape but the ordinates are reduced roughly by an order of magnitude; in that case, the framed tube wall is likely to buckle as a plate, which requires four hinges to form on some columns lines and three on others (see Fig. 2c of Bažant and Zhou). Such a buckling mode is suggested by photographs of flying large fragments of the framed-tube wall, which show rows of what looks like broken-off plastic hinges.

The mass of columns is assumed to be lumped, half and half, into the mass of the upper and lower floors, while the columns are massless. This avoids calculating the elastic waves propagating along the collapsing columns between the floors (which cannot interfere with buckling because their travel times are orders-of-magnitude shorter).

Deceleration and Acceleration During the Crushing of One Story. The two intersections of the horizontal line $F = gm(z)$ with the curve $F(u)$ seen in Fig. 3 and 4a (top) are equilibrium states (there is also a third equilibrium state at intersection with the vertical line of rehardening upon contact). But any other state on this curve is a transient dynamic state, in which the difference from the line $F = gm(z)$ represents the inertia force that must be generated by acceleration or deceleration of the block of tower mass $m(z)$ above level z of the top floor of the story.

Before impact by upper part, the columns are in equilibrium, i.e. $F(u_0) = gm(z)$, where u_0 = initial elastic shortening of columns under weight $gm(z)$ (about $0.0003h$). At impact, the initial condition for subsequent motion is velocity $v_0 = \dot{u}(u_0) \approx v_i$ = velocity of the impacting block of upper part of tower. Precisely, from balance of linear momentum upon impact, $v = m(z)/[m(z) + m_F]$, but this is only slightly less than v_i because $m_F \ll m(z)$ (m_F is the mass of the impacted upper floor).

When $F(u) \neq gm(z)$, the difference $F(u) - gm(z)$ causes deceleration of mass m_z if positive (ΔF_d in Fig. 3) and acceleration if negative (ΔF_a in Fig. 3). The equation of motion of mass m(z) during the crushing of one story (or one group of stories, in the case of multi-story buckling) reads as follows:

$$\ddot{u} = g - F(u)/m(z) \tag{2}$$

where z = constant = coordinate of the top floor of the story, and superior dots denote derivatives with respect to time t. So, after impact, the column resistance causes mass $m(z)$ to decelerate, but only until point u_c at which the load-deflection diagram intersects the line $F = gm(z)$ (Figs. 3, 4a). After that, mass $m(z)$ accelerates until the end of column crushing.

If the complete function $F(u)$ is known, then calculation of the motion of the upper part of tower from (2) is easy (to calculate this function precisely is a formidable problem, but an upper bound curve is easy to figure out from plastic hinges, Bažant and Zhou 2002). Examples of accurately computed velocity $v = \dot{u}$ from Eq. (2) for various load-displacement diagrams graphically defined in the top row of Fig. 4a are shown in rows 2 and 3 of Fig. 4a,b,c.

Energy Criterion of Progressive Collapse Trigger. The energy loss of the columns up to displacement u is

$$\Phi(u) = \int_{u_0}^{u} [F(u') - gm(z)] \, \mathrm{d}u' = W(u) - gm(z)u \tag{3}$$

$$W(u) = \int_{u_0}^{u} F(u') \, \mathrm{d}u' \tag{4}$$

where z = constant = column top coordinate, $W(u)$ = energy dissipated by the columns = area under the load-displacement diagram (Fig. 3) and $-gm(z)u$ = gravitational potential change causing an increment of kinetic energy of mass $m(z)$. Note that, since the possibility of unloading (Fig. 4c top) can be dismissed, $W(u)$ is path-independent and thus can be regarded, from the thermodynamic viewpoint, as the internal energy, or free energy, for very fast (adiabatic), or very slow (isothermal) deformations, and thus $\Phi(u)$ represents the potential energy loss. If $F(u) < gm(z)$ for all u, $\Phi(u)$ continuously decreases. If not, then $\Phi(u)$ first increases and then decreases during the collapse of each story. Clearly, collapse will get arrested if and only if the kinetic energy does not suffice for reaching the interval of accelerated motion, i.e., the interval of decreasing $\Phi(u)$, i.e., (Fig. 4 right column). So, the crushing of columns within one story will get arrested before completion (Fig. 4c) if and only if

$$\mathcal{K} < W_c \tag{5}$$

where $W_c = \Phi(u_c) = W(u_c) - gm(z)u_c$ = net energy loss up to u_c during the crushing of one story, and $\mathcal{K}$ = kinetic energy of the impacting mass $m(z)$. This is the criterion of preventing progressive collapse to begin (Fig. 4c). Its violation triggers progressive collapse.

Graphically, this criterion means that $\mathcal{K}$ must be smaller than the area under the load-deflection diagram lying above the horizontal line $F = gm(z)$ (Fig. 3 and 4 right column). If this condition is violated, the next story will again suffer an impact and the collapse process will be repeated.

The next story will be impacted with higher kinetic energy if and only if

$$W_g > W_p \tag{6}$$

where $W_g = gm(z)u_f$ = loss of gravity when the upper part of tower is moved down by distance u_f, $u_f = (1-\lambda)h$ = final displacement at full compaction, and $W_p = W(u_f) = \int_0^{u_f} F(u)\mathrm{d}u$ = area under the complete load-displacement curve $F(u)$ (Fig. 3). This is the criterion of accelerated collapse.

For the case of WTC towers, it was estimated by Bažant and Zhou (2002) that $\mathcal{K} \approx 8.4W_p \gg W_p$ for the story where progressive collapse initiated. W_g was greater than W_p by an order of magnitude, which guaranteed acceleration of collapse from one story to the next.

Some investigators have been under the mistaken impression that collapse cannot occur if (because of safety factors used in design) the weight mg of the upper part is less than the load capacity F_0 of the floor. This led them to postulate various strange ideas (such as "fracture wave" and planted explosives). However, the criterion in (5) makes it clear that this impression is erroneous. If this criterion is violated, there is (regardless of F_0) no way to deny the inevitability of progressive collapse driven *only* by gravity.

Options for Transition to Global Continuum Model. One option would be finite element simulation based on the traditional homogenization of heterogenous microstructure of the tower, in which the load-displacement curve $F(u)$ in Fig, 3 would be converted to an averaged stress-strain curve $\sigma(\epsilon)$ by setting $\epsilon = u/h$ and $\sigma = F/A$) (A = cross-section area of the tower). However, the stress-strain relation delivered by this standard homogenization approach would exhibit strain softening. This would lead to an ill-posed dynamic problem, whose mathematical solution exists but is physically wrong (Bažant and Belytchko 1985, Bažant and Cedolin 2003, sec. 13.1). To obtain a well-posed formulation, it would be necessary to regularize the initial-boundary value problem by introducing a nonlocal formulation (Bažant and Jirásek 2004, Bažant and Cedolin 2003, chapter 13) with a characteristic length equal to the story height h (such regularization was forgotten in the "fracture wave" theory, proposed as an alternative for modeling WTC collapse]. But the nonlocal approach is relatively complex to program and gradual strain softening need not be modelled because only the total energy release per story is important (as demonstrated in rows 2 and 3 of Fig. 4 by equivalence of velocity diagrams).

Because of the clearly defined periodic microstructure of stories in the tower, there is another, more effective, option: *non-softening energetically equivalent* characterization of discrete elements—the individual failing stories. This option is pursued next. It corresponds to non-standard homogenization, in which in which the aim is not homogenized stiffness but homogenized energy dissipation (this approach is analogous to the energy equivalent transition in van der Waals theory of gas-liquid phase changes, and the energy equivalence is also analogous to the crack band model for softening distributed damage, Bažant and Cedolin 2003, Bažant and Jirásek, 2004).

Energetically Equivalent Mean Crushing Force. For the purpose of continuum smearing of a tower with many stories, the actual load-displacement diagram $F(z)$ (curve OABC in Fig. 2a) can be replaced by a simple energetically equivalent diagram, represented by the horizontal line $F = F_c$. Here F_c is the mean crushing force (or resistance) at level z, such that the dissipated energy per story, represented by the rectangular area under the horizontal line $F = F_c$, be equal to the total area W_p under the actual load-displacement curve OABC; i.e.,

$$F_c = \frac{W_p}{u_f} \;=\; \frac{1}{u_f}\int_0^{u_f} F(u)\,\mathrm{d}u \tag{7}$$

The energy-equivalent replacement avoids unstable snap-through (Bažant and Cedolin 2003) (and is analogous to what is in physics of phase transitions called the Maxwell line). Although the dynamic $u(t)$ history for the replacement F_c is not the same as for the actual $F(u)$, the final values of displacement u and velocity $\dot{u}$ at the end of crushing of a story are exactly the same, as shown in the exactly calculated diagrams in rows 2 and 3 of Fig. 4. So the replacement has no effect on the overall change of velocity v of the collapsing story from the beginning to the end of column crushing (Fig. 4), i.e, from $u = 0$ to $u = u_f$ (as long as F_c is not large enough to arrest the downward motion). F_c may also be regarded as the mean energy dissipated per unit height of the tower, which has the physical dimension of force.

Note that it would be slightly more accurate not to include the minuscule elastic strain-energy portion of W_p in integral (7), i.e., replace the lower limit 0 with u_0. But then, instead of constant F_c, we would need to consider an elastic-perfectly plastic force-displacement relation. The stiff and short elastic stress rise from $u = 0$ to u_0 (Fig. 4) would produce elastic waves which would create an unnecessary noise in the response and complicate numerical simulation).

One-Dimensional Continuum Model for Crushing Front Propagation

Detailed finite element analysis simulating plasticity and break-ups of all column and beams, and the flight and collisions of broken pieces, would be extremely difficult, as well as unsuited for extracting the basic general trends. Thus it appears reasonable to make four simplifying hypotheses: (i) The only displacements are vertical and only the mean of vertical displacement over the whole floor needs to be considered. (ii) Energy is dissipated only at the crushing front (this implies that the blocks in Fig. 2 may be treated rigid, i.e., deformations of the blocks away from the crushing front may be neglected). (iii) The relation of resisting normal force F transmitted by columns of each floor to the relative displacement u between two adjacent floors obeys a known load-displacement diagram (Fig. 4, terminating with a specified compaction ratio λ (which must be adjusted to take into account lateral shedding of a certain known fraction of rubble outside the tower perimeter). (iv) The stories are so numerous, and the collapse front traverses so many stories, that a continuum smearing (i.e., homogenization) gives a sufficiently accurate overall picture.

The one-dimensionally idealized progress of collapse of a tall building (of initial height H) is idealized in Fig. 2, where ζ, η = coordinates measured from the initial and current tower top, respectively; $z(t), y(t)$ = coordinates ζ and η of the crushing front at time t (ζ is the Lagrangian coordinate of material points in the sense of finite strain theory, while y is measured from the moving top of building). The initial location of the first floor crashing into the one below is at $\zeta = z = z_0 = y_0$. The resisting force F and compaction ratio λ are known functions of z. A and C label the lower and upper undisturbed parts of tower, and B the zone of crushed stories compacted from initial thickness s_0 to the current thickness

$$s(t) = \int_{\zeta=z_0}^{z(t)} \lambda(\zeta) \mathrm{d}\zeta \tag{8}$$

When μ = constant, $s(t) = \lambda(z(t) - z_0)$ where $z(t) - z_0$ is the distance that the crushing front has traversed through the tower at time t. The velocity of the upper part of tower is

$$v(t) = [1 - \lambda(z)]\dot{z}(t) \tag{9}$$

First it needs to be decided whether crushed zone B will propagate down or up through the tower. The equation of motion of zone B requires that

$$F_1 - F_2 = \lambda s_0 [\mu g - (\mu v)^{\cdot}] \tag{10}$$

This expression must be positive if the zone B is falling slower than a free fall, which is reasonable to expect and is confirmed by the solution to be given. Therefore always $F_2 < F_1$. So, neither upward, nor two-sided simultaneous, propagation of crushing front is possible.

Note that a front propagating intermittently up and down would nevertheless be found possible if $F_c(z)$ were considered to be a random (autocorrelated) field. In that case, short intervals Δt may exist in which the difference $F_{c1} - F_{c2}$ of random F_c values at bottom and top of crushed block B would exceed the right-hand side of Eq. (10). During those short intervals, crush-up would occur instead of crush-down, more frequently for a larger coefficient of variation. The greater the value of s_0, the larger the right-hand side of (10), and thus the smaller the chance of crush-up. So, random crush-up intervals could exist only at the beginning of collapse, when s_0 is still small enough. Stochastic analysis, however, is beyond the scope of this study.

The phase of downward propagation of the front will be called the *crush-down* phase, or Phase I (Fig. 4b). After the lower crushing front hits the ground, the upper crushing front of the compacted zone can begin propagating into the falling upper part of tower (Fig. 4d). This will be called the *crush-up* phase, or Phase II (it could also be called the 'demolition phase', because demolitions of buildings are usually effected by explosive cutter charges placed at the bottom).

Let $\mu = \mu(\zeta)$ = initial mass density at coordinate ζ = continuously smeared mass of undisturbed tower per unit height. The mass density of the compacted zone B is $mu(z)/\lambda(z)$ $(> \mu)$. However, a correction must be made for the fraction κ_{out} of the mass that is being lost at the crushing front, ejected into the air outside the perimeter of the tower. During crash-down, the ejected mass alters the inertia and weight of the moving compacted part B, which requires a correction to $m(z)$, while during crash-up no correction is needed because part B is not moving. Accordingly, we adjust the definition of the inertial mass of the tower above level z in the crush-down phase as follows:

$$\text{For } z > z_0: \qquad m(z) = m(z_0) + \int_{z_0}^{z} (1 - \kappa_{out})\mu(\zeta)\mathrm{d}\zeta, \qquad m(z_0) = \int_0^{z_0} \mu(\zeta)\mathrm{d}\zeta \tag{11}$$

No adjustment is needed for the crush-up phase because block B of compacted rubble does not move with C but is is stationary.

Differential Equations of Progressive Collapse or Demolition

The differential equations for $z(t)$ and $y(t)$ can be obtained from dynamic free body diagrams (Fig. 4h). In the crush-down phase, the compacted zone B and the upper part A of tower move together as one rigid body accreting mass, with combined momentum $(1 - \lambda)m(z)\dot{z}$. The negative of the derivative of this momentum is the upward inertia force. Additional vertical forces are weight $m(z)g$ downward, and resistance $F_c(z)$ upward. The condition of dynamic equilibrium according to d'Alembert principle yields the following differential equation for compaction front propagation in the crush-down phase I of progressive collapse:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{ m(z)[1 - \lambda(z)]\frac{\mathrm{d}z}{\mathrm{d}t} \right\} - m(z)g = -F_c(z) \qquad \text{(crush-down)} \tag{12}$$

For the special case of $\lambda = F_c = \kappa_{out} = 0$, Eq. (12) reduces to $(z\dot{z})^{\cdot} = z$ (the numerical solution for this special case was presented by Kausel, 2001).

The initial conditions for the crush-down phase I are $z = z_0$ and $\dot{z} = 0$. Downward propagation will start if only if

$$m(z_0)g > F_c(z_0) \tag{13}$$

In the crush-up phase, the crushing front at $\eta = y$ is moving up with velocity $\lambda(y)\dot{y}$, and so the downward momentum of part C is $m(y)[1 - \lambda(y)]\dot{y}$. Downward acceleration of part C is opposed by upward inertia force

$$F_i^C = -\{m(y)[1-\lambda(y)]\dot{y}\}^{\cdot} \tag{14}$$

By contrast to the crush-down phase, the compacted zone B with accreting mass is not moving with part C but is now stationary (Fig. 4d), and this makes a difference. During every time increment dt, the momentum

$$\mathrm{d}p = [\mu(y)(\dot{y}\mathrm{d}t)][1-\lambda(y)]\dot{y} \tag{15}$$

of the infinitesimal slice $\mathrm{d}y = \dot{y}\mathrm{d}t$ at the crushing front gets reduced to 0 ($\dot{y} < 0$). So, the stationary part B is subjected to downward inertia force (Fig. 4g):

$$F_i^B = \mathrm{d}p/\mathrm{d}t = \mu(y)[1-\lambda(y)]\dot{y}^2 \tag{16}$$

(this is a similar phenomenon as, in the kinetic theory of gases, the pressure of gas molecules hitting a wall). As a reaction, the same force acts upward from part B onto part C. Adding also the force of gravity (and noting that $\dot{y} < 0, \ddot{y} < 0$), the dynamic equilibrium of part C as a free body requires that $F_i^B - F_i^C - m(y)g + F_c = 0$. This yields the following differential equation for compaction front propagation in the crush-up phase of progressive collapse:

$$m(y)\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[[1-\lambda(y)]\frac{\mathrm{d}y}{\mathrm{d}t}\right] + g\right\} = F_c(y) \qquad \text{(crush-up)} \tag{17}$$

For the special case of $\lambda = F_c = 0$ and constant μ (for which $m = \mu y$), Eq. (17) reduces to $\ddot{y} = -g$, which is the equation of free fall of a fixed mass.

For the special case when only λ is constant while $F_c(y)$ and $\mu(y)$ vary, Eq. (17) reduces to

$$\ddot{y} = -\tilde{g}(y), \quad \tilde{g}(y) = [g - F_c(y)/m(y)]/(1-\lambda) \tag{18}$$

This is equivalent to a fall under variable gravity acceleration $\tilde{g}(y)$. Obviously, the collapse will accelerate (for $\lambda \neq 0$) only as long as $\tilde{g} > 0$, i.e., if condition (13) is satisfied. Since $\lim_{y\to 0} m(y) = 0$, this condition will always become violated before collapse terminates (unless $F_c = 0$), and so the collapse must decelerate at the end.

For $F_c > 0$, the tower can in fact never collapse totally, i.e, $y = 0$ cannot be attained. To prove it, consider the opposite, i.e. $y \to 0$. Then $\ddot{y} = C/y$ where $C = F_c/\mu(1-\lambda) =$ constant > 0; hence $(\dot{y}^2)^{\cdot} = 2\dot{y}\ddot{y} = 2C\dot{y}/y$, the integration of which gives $\dot{y}^2 = 2C\ln(y/C_1)$ where C_1 is a constant. The last equation cannot be satisfied for $y \to 0$ because the left-hand side ≥ 0 while the right-hand side $\to -\infty$; Q.E.D.

As the rubble height approaches its final value, i.e. for $\lim_{\dot{y}\to 0} = y_f (> 0)$, the values of m, λ, F_c are nearly constant, and so $\ddot{y} = (F_c/m - g)/(1-\lambda) = C_0 =$ constant [> 0, which is again condition (13)]. Hence, $\ddot{y} = C_0$, which gives $y(t) - y_f = C_0(t-t_f)^2$. So, if $F_c > 0$, the collapse history $y(t)$ will terminate asymptotically as a parabola at some finite height y_1 and finite time t_f.

For a more detailed simulation of collapse, it would be possible to use for each floor Eq. (2) for motion within each floor, or introduce into Eq. (12) and (17) a function $F_c(z)$ varying within each story height as shown by the actual response curves in Figs. 4 and 5. This would give a fluctuating response with oscillations superposed on the same mean trend of $z(t)$ or $y(t)$ as that for smooth $F_c(z)$. Extremely small time steps would be needed in this case.

The fact that F_c is smaller in the heated story than in the cold stories may be taken into account by reducing $F_c(z)$ within a certain interval $z \in (z_0, z_1)$.

The initial conditions for the crush-up phase II are $y = y_0 = z_0$ and a velocity $\dot{y}$ equal to the terminal velocity of the crush-down phase. For a demolition, triggered at the base of building, the initial conditions are $y = y_0$ and $\dot{y} = 0$, while $F_c = 0$ for the y value corresponding to the ground story height.

For trigger by an explosion or vertical impact, the present formulation might be used with an initial condition consisting of a certain finite initial velocity v_0 caused by the explosion. In that case, $\mathcal{K}$ in collapse trigger criterion (5)may be replaced by energy imparted by the explosion.

Dimensionless Formulation

To convert the formulation to a dimensionless form, note that the solution can be considered to be a function of 2 coordinates, t and z (or y), and 6 independent parameters, $H, z_0, g, F_c, \mu(z), \lambda(z)$, and involves involves 3 independent dimensions, the mass, length and time. According to the Vashy-Buckingham theorem, the solution must depend on only $7 + 2 - 3 = 6$ dimensionless independent parameters, of which 2 are the dimensionless time and spatial coordinate. They may be chosen as follows:

$$\begin{aligned} &\tau = t\sqrt{g/H}, \quad Z = z/H \text{ or } Y = y/H, \quad Z_0 = z_0/H = y_0/H \\ &\bar{F}_c(Z) = F_c(z)/Mg, \quad \bar{m}(Z) = m(z)/M, \quad \lambda = \lambda(Z) \end{aligned} \tag{19}$$

where $M = m(H)$ = total mass of the tower. After transformation to these variables, the differential equations of the problem, Eqs. (12) and (17), take the following dimensionless forms:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{ [1-\lambda(Z)]\, \bar{m}(Z)\, \frac{\mathrm{d}Z}{\mathrm{d}\tau} \right\} - \bar{m}(Z) = -\bar{F}_c(Z) \qquad \text{(crush-down)} \tag{20}$$

$$\bar{m}(Y)\left\{ \frac{\mathrm{d}}{\mathrm{d}\tau}\left[[1-\lambda(Y)]\, \frac{\mathrm{d}Y}{\mathrm{d}\tau} \right] + 1 \right\} = \bar{F}_c(Y) \qquad \text{(crush-up)} \tag{21}$$

The dimensionless form of the initial conditions is obvious.

In the special case of constant μ and λ, we have $\bar{m}(Z) = Z$, $\bar{m}(Y) = Y$, and the foregoing dimensionless differential equations take the form:

$$(1-\lambda)(Z\ddot{Z} + \dot{Z}^2) - Z = -\bar{F}_c(Z) \qquad \text{(crush-down)} \tag{22}$$

$$(1-\lambda)Y\ddot{Y} + Y = \bar{F}_c(Y) \qquad \text{(crush-up)} \tag{23}$$

Numerical Solution and Parametric Study

Eq. (12) may be converted to a system of 2 first-order differential equations of the form $\dot{z} = x$ and $\dot{x} = F(x, z)$, with prescribed values of z and x as the initial conditions. This system can be easily solved by some efficient standard numerical algorithm, such as the Runge-Kutta method. The same is true for Eq. (17).

The diagrams in Fig. 6 present the collapse histories computed for the approximate parameters of WTC (heavy solid curves) and for modified values of these parameters. For comparison, the curve of free fall from the tower top is shown in each diagram as the leftmost curve. The transition from the crush-down phase I to the crush-up phase II is marked in each diagram (except one) by a horizontal line. The parameter values used for calculation are listed in each diagram. There were chosen as the typical values for the WTC and their variations. $\bar{W}_f$ denotes the mean of a linearly varying crushing energy W_f. Since the first story to collapse was heated, the value of F_c within the interval of z corresponding to the height of that story was reduced to one half. Fig. 7 shows separately the histories of tower top coordinate for the crush-up phase alone, which is the case of demolition. Four characteristics of the plots of numerical results in Fig. 6 and 7 should be noticed:

- Varying the building characteristics, particularly the crushing energy W_f per story, makes a large enough difference in response to be easily detectable by the monitoring of collapse.
- The effect of crushing energy W_f on the rate of progressive collapse is much higher than the effect of compaction ratio λ or specific mass μ. This means that these two parameters need not be estimated very accurately in advance of inverse analysis.
- For the structural system such as WTC, the energy required to arrest the collapse after a drop of only one or several stories (Fig. 6e) would have to be an order of magnitude higher than the energy dissipation capacity of the structural system used in WTC .
- For the typical WTC characteristics, the collapse takes about 10.8 s (Fig. 6 top left), which is not much longer (precisely only 17% longer) than the duration of free fall in vacuum from tower top to the ground, which is 9.21 s (the duration of 10.8 s is within the range of Bažant and Zhou's, 2002, crude estimate). For all of the wide range of parameter values considered in Fig. 6, the collapse takes less than about the double of free fall duration.

The last two points confirm Bažant and Zhou's (2002) observations about collapse duration made on the basis of initial kinetic energy and without any calculation of collapse history.

What Can We Learn?—Proposal for Monitoring Demolitions

We have seen that the main unknown in predicting cohesive collapse is the mean energy dissipation W_f per story. The variable $\mu(z)$ is known from the design, and the contraction ratio $\lambda(z)$ can be reasonably estimated from Eq. (1) based on observing the rubble heap after collapse. But a theoretical or computational prediction of F_c is extremely difficult and fraught with uncertainty. Precise collapse observations are required.

Eqs. (12) and (17) show that $F_c(z)$ can be evaluated from the monitoring of motion history $z(t)$ and $y(t)$, provided that $\mu(z)$ and $\lambda(z)$ are known. Such information can, in theory, be extracted from a high-speed camera record of the collapse. Approximate information could have been extracted from a regular video of collapse if the moving part of the towers of WTC were not shrouded in a cloud of dust and smoke. Thus, despite thousands of photographic records, nothing can be learned from WTC.

However, valuable information on the energy dissipation capability of various types of structural systems could be extracted by monitoring demolitions. During the initial period of demolition, the precise history of motion of building top could be determined from a video of collapse. After the whole building disappears in dust cloud, various remote sensing techniques could be used. For example, one could follow through the dust cloud the motion of sacrificial radio transmitters. Or one could install sacrificial accelerometers monitored by real-time telemetry. From the acceleration record, the $y(t)$-history could be integrated.

Therefore, monitoring of demolitions is proposed as a means of learning about the energy absorption capability of various structural systems.

Usefulness of Varying Demolition Mode. Ronan Point apartments, the Oklahoma City bombing, etc., demonstrate that only a vertical slice of building may undergo progressive collapse, while the remainder of building stands. Such a collapse is truly a three-dimensional problem, much harder to analyze, but some cases might allow adapting the present one-dimensional model as an approximation. For example, in Ronan Point apartments, energy was dissipated not only by vertical crushing of stories, but also by shearing successive floor slabs from their attachments to columns on the side of the collapsing stack of rooms. The present model seems usable if the energy dissipated by shearing is added to the crushing energy F_c, and if the rotational kinetic energy of floor slabs whose fall is hindered on one side by column attachments is taken into account. Such a generalization of the present model could be calibrated by comparing data from two different demolition modes: 1) the usual mode, in which the building is made to collapse symmetrically, and 2) another mode in which first only a vertical slice of building (e.g., one stack of rooms) is made to collapse by asymmetrically placed cutter charges. Many variants of this kind may be worth studying.

Complex Three-Dimensional Situations. Situations such as stepped tall buildings call for three-dimensional analysis. Large-scale finite-strain computer simulation tracking the contacts of all the pieces of crushing floors and columns could in principle do the job but would be extraordinarily tedious to program and computationally demanding. The present analysis would be useful for calibrating such a computer program.

Massive Structures. Progressive collapse is not out of question even for the massive load-bearing concrete cores of the tallest recent skyscrapers, as well as for tall bridge piers and tall towers of suspension or cable-stayed bridges (that such a collapse mode is a possibility is documented, e.g., by the collapses of Campanile in Venice and Civic Center tower in Pavia). Although progressive collapse of the modern massive piers and towers would be much harder to initiate, a terrorist attack of sufficient magnitude might not be inconceivable. Once a local damage causes a sufficient downward movement of the superior part of structure, it cannot be stopped. The question is, for instance, whether it might be within the means of a terrorist to cause, e.g., formation and slipping of an inclined crack band typical of compression fracture of concrete. In this regard, note that the size effect in compression fracture (Cusatis and Bažant 2006) would assist a terrorist.

Alternative Formulations, Extensions, Ramifications

Alternative Derivation. A more elementary way to derive the differential equation for the crush-up phase is to calculate first the normal force $N(\eta)$ (positive if tensile) in cross section of any coordinate $\eta \in (0, y)$ (Fig. 4h). The downward velocity of block C is $v = [1 - \lambda(y)]\dot{y}$, and its acceleration is opposed by inertia force $[1 - \lambda(y)]\ddot{y}m(\eta)$. The downward gravity force on this block is $gm(\eta)$. From dynamic equilibrium, the normal force $N(\eta)$ (positive if tensile), acting at the lower face η of this block, is:

$$N(\eta) \;=\; -[1 - \lambda(y)]\; \ddot{y}\; m(\eta) \;+ g\; m(\eta) \qquad (24)$$

For the crushing front, $\eta = y$, this must be equal to the crushing force, i.e., $N(y) = -F_c(y)$. This immediately verifies Eq. 17.

For the crush-down phase, the same expression holds for the cross section force $N(\zeta)$. However, in the dynamic equilibrium condition of block C, one must add upward inertia force $\mu(z)\dot{z}^2$

needed to accelerate from 0 to $\dot{z}$ the mass that is accreting to block C per unit time. This then verifies Eq. 12.

Potential and Kinetic Energies. An energy based formulation is useful for various approximations, numerical algorithms and bounds. It is slightly complicated by the accretion of mass to the moving block and the dissipation of energy by crushing force F_c.

Consider first the crush-down phase. Since unloading of columns does not occur, a potential Π can be defined as the gravitational potential minus the work of F_c. Its rate is:

$$\frac{\mathrm{d}\Pi(t)}{\mathrm{d}t} = \{F_c[z(t)] - gm[z(t)]\}v(t) \tag{25}$$

Due to accretion of mass to the moving block, its kinetic energy $m(z)v^2/2$ in increased by the kinetic energy due to accelerating every infinitesimal slice $\mathrm{d}z = \dot{z}\mathrm{d}t$ of mass $m'(z)(\dot{z}\mathrm{d}t)$ to velocity v. This means that kinetic energy increment $\frac{1}{2}[m'(z)(\dot{z}\mathrm{d}t)]v^2$ is added during every time increment $\mathrm{d}t$. So, the rate of added kinetic energy is $\frac{1}{2}m'(z)\dot{z}v^2$, and the overall rate of change of kinetic energy $\mathcal{K}$ is

$$\frac{\mathrm{d}\mathcal{K}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}m[z(t)]v^2(t)\right\} + \frac{1}{2}m'(z)v^2(t)\frac{\mathrm{d}z(t)}{\mathrm{d}t} \tag{26}$$

where $m'(z) = \mathrm{d}m(z)/\mathrm{d}z$, which differs from $\mu(z)$ only at locations where κ_{out} has an effect. Conservation of energy requires the sum of the last two energy rates to vanish. This condition yields:

$$m(z)v\dot{v} + \tfrac{1}{2}m'(z)(z)v^2\dot{z} + \tfrac{1}{2}m'(z)(z)\dot{z}v^2 + [F_c(z) - gm(z)]v = 0 \tag{27}$$

Dividing this equation by mass velocity v and setting $v = (1-\lambda)\dot{z}$, we find that Eq. (17) ensues. This verifies correctness of the foregoing energy expressions for the crush-down phase.

For the crush-up phase, the rate of energy potential is

$$\frac{\mathrm{d}\Pi(t)}{\mathrm{d}t} = \{gm[y(t)] - F_c[y(t)]\}v(t) \tag{28}$$

In formulating the kinetic energy, there is a difference from crush-down: The mass of each infinitesimal slice $\mathrm{d}y = \dot{y}\mathrm{d}t$ is during $\mathrm{d}t$ decelerated from velocity v to 0, removed from the moving block C, and added to the stationary block B. By analogous reasoning, one gets for the kinetic energy rate the expression:

$$\frac{\mathrm{d}\mathcal{K}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}m[y(t)]v^2(t)\right\} - \frac{1}{2}\mu[y(t)]v^2(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t} \tag{29}$$

where $\mu(y) = m'(y)$. Energy conservation dictates that the sum of the last two energy rate expressions must vanish, and so

$$m(y)v\dot{v} + \tfrac{1}{2}\mu(y)v^2\dot{y} - \tfrac{1}{2}\mu(y)\dot{y}v^2 + [gm(z) - F_c(z)]v = 0 \tag{30}$$

After division by $v = (1-\lambda)\dot{y}$, Eq. (12) for the crush-up phase is recovered. This agreement verifies the correctness of the foregoing energy rate expressions.

Lagrange equations of motion or Hamilton's principle (Flügge 1962) are often the best way to analyze complex dynamic systems. So why hasn't this approach been followed?—Because these equations are not valid for systems with variable mass. Although various special extensions to such systems have been formulated, they are complicated and depend on the particular type of system (e.g., Pesce 2003).

Solution by Quadratures for Constant λ and μ, and $\kappa_{out} = 0$. In this case, which may serve as a test case for finite element program, Eq. (12) for the crush-down phase takes the form:

$$f\ddot{f} + \dot{f}^2 - Qf = -P \qquad \text{or} \qquad (f\dot{f})^{\cdot} = Qf - P \tag{31}$$

Here $Q = 1/(1-\lambda)$, $P(t) = F_c/\mu(1-\lambda)gH$, $F_c = F_c[z(t)]$, $f = f(t) = z(t)/H$; and the superior dots now denote derivatives with respect to dimensionless time $\tau = t\sqrt{g/H}$. Let $\varphi = f^2/2$. Then $\dot{\varphi} = f\dot{f}$ and

$$\ddot{\varphi} = Q\sqrt{2\varphi} - P \tag{32}$$

$$(\dot{\varphi}^2)^{\cdot} = 2\dot{\varphi}\ddot{\varphi} = 2(Q\sqrt{2\varphi} - P)\dot{\varphi} \tag{33}$$

$$\int \mathrm{d}(\dot{\varphi}^2) = \int 2(Q\sqrt{2\varphi} - P)\mathrm{d}\varphi \tag{34}$$

$$\dot{\varphi} = \left(\tfrac{4}{3}Q\sqrt{2}\varphi^{3/2} - 2P\varphi + C\right)^{1/2} \tag{35}$$

$$\tau - \tau_0 = \int_{\varphi(\tau_0)}^{\varphi(\tau)} \left(\tfrac{4}{3}Q\sqrt{2}\varphi^{3/2} - 2P\varphi + C\right)^{-1/2} \mathrm{d}\varphi \tag{36}$$

The second equation was obtained by multiplying the first by $2\dot{\varphi}$, and Eq. 35 was integrated by separation of variables; C and τ_0 are integration constants defined by the initial conditions. The last equation describes the collapse history parametrically; for any chosen φ, it yields the time as $t = z\sqrt{H/g}$ or $y\sqrt{H/g}$ where z or $y = H\sqrt{2\varphi}$.

Eq. (12) for the crush-up phase with constant μ and λ takes the form:

$$f\ddot{f} + Qf = P \tag{37}$$

Multiplying this equation by $\dot{f}/f$ and noting that $\dot{f}\ddot{f} = \frac{1}{2}(\dot{f}^2)^{\cdot}$ and $\dot{f}/f = (\ln f)^{\cdot}$, one may get the solution as follows:

$$(\dot{f}^2)^{\cdot} = 2(P\dot{f}/f - Q\dot{f}) \tag{38}$$

$$\dot{f}^2 = 2(P\ln f - Qf) + C \tag{39}$$

$$\mathrm{d}f = [2(P\ln f - Qf) + C]^{1/2}\mathrm{d}\tau \tag{40}$$

$$\tau - \tau_0 = \int_{f(\tau_0)}^{f(\tau)} [2(P\ln f - Qf) + C]^{-1/2}\mathrm{d}\tau \tag{41}$$

Effect of Elastic Waves. Note that Eqs. (12) and (17) do not represent elastic stiffness and thus do not model wave propagation. Elastic waves are ignored because the maximum shortening of story height for which the columns remain elastic is only about 0.4 mm. If elastic response were incorporated, a much faster elastic wave, with step wavefront equal to F_0, would be found to emanate from the crushing front when each floor is hit, propagate down the tower, reflect from the ground, etc. But the damage to the tower would be almost nil, the subsequent propagation of the crushing front would be almost unaffected by the precursor elastic waves, and the solution would get more complicated.

Analogous problem—Crushing of Foam. One-dimensional crushing of rigid foam by impact can be solved from the present differential equation for the crush-down phase. To this end, the top part of tower needs to be replaced by a rigid impacting object of mass equivalent to $m(z_0)$, whose initial velocity is assigned as the initial condition at $t = 0$. Compared to inertia forces, normally one can neglect gravity (i.e., set $g = 0$).

Implications and Conclusions

1. If the total (internal) energy loss during the crushing of one story (representing the energy dissipated by the complete crushing and compaction of one story, minus the loss of gravity potential during the crushing of that story), exceeds the kinetic energy impacted to that story, collapse will continue to the next story. This is the criterion of progressive collapse trigger (Eq. 5). If it is satisfied, there is no way to deny the inevitability of progressive collapse driven by gravity *alone* (regardless of by how much the combined strength of columns of one floor may exceed the weight of the part of tower above that floor).

2. One-dimensional continuum idealization of progressive collapse is amenable to a simple analytical solution which brings to light the salient properties of the collapse process. The key idea is not to use classical homogenization, leading to a softening stress-strain relation necessitating nonlocal finite element analysis, but to characterize the stories by an energetically equivalent snapthrough.

3. Distinction must be made between crush-down and crush-up phases, for which the crushing front of a moving block with accreting mass propagates into the stationary stories below, or into the moving stories above, respectively. This leads to a second-order nonlinear differential equation for propagation of the crushing front in the crush-down phase, or in the subsequent crush-up phase.

4. The mode and duration of collapse of WTC towers are consistent with the derived model, but nothing more can be learned because the motion was obstructed from view by a cloud of dust and smoke.

5. The present idealized model allows simple inverse analysis which can yield the crushing energy per story and other properties of the structure from a precisely recorded history of motion during collapse. From the crushing energy, one can infer the collapse mode, e.g., single-story or multi-story buckling of columns.

6. It is proposed to monitor the precise time history of displacements in building demolitions—for example by radio telemetry from sacrificial accelerometers, or high-speed optical camera—and to engineer different modes of collapse to be monitored. This should provide invaluable information on energy absorption capability of various structural systems, needed for assessing the effects of explosions, impacts, earthquake, and terrorist acts.

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List of Figures

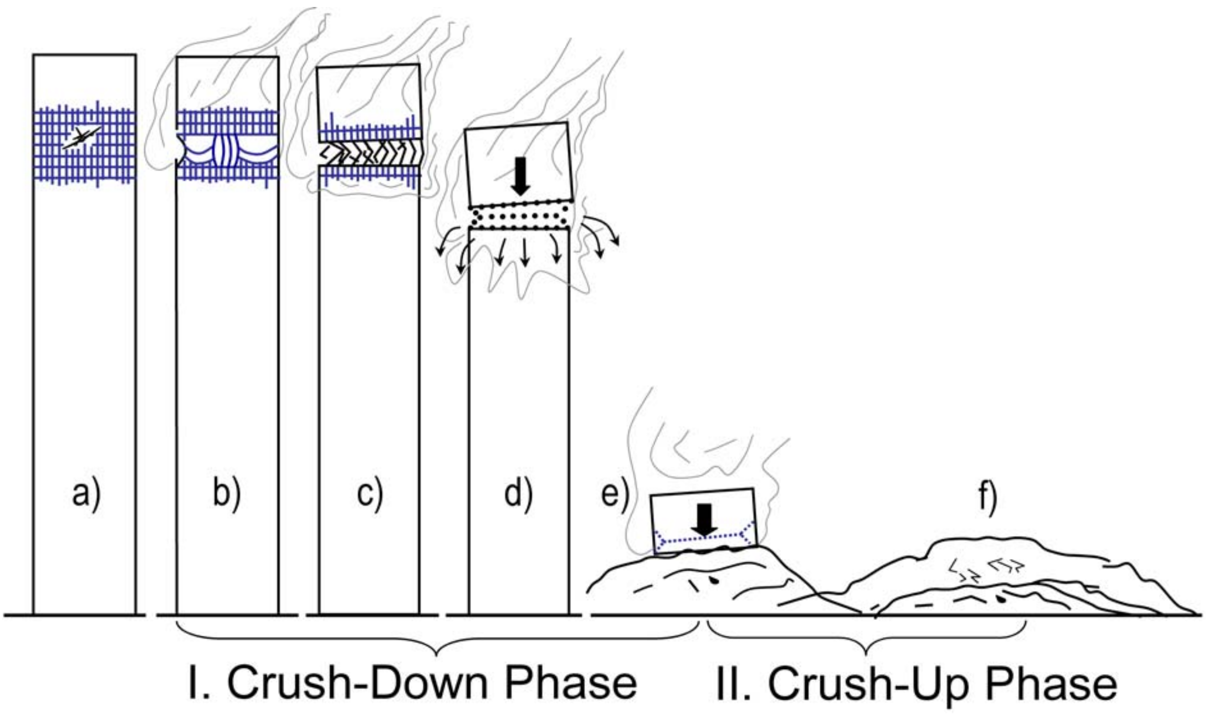
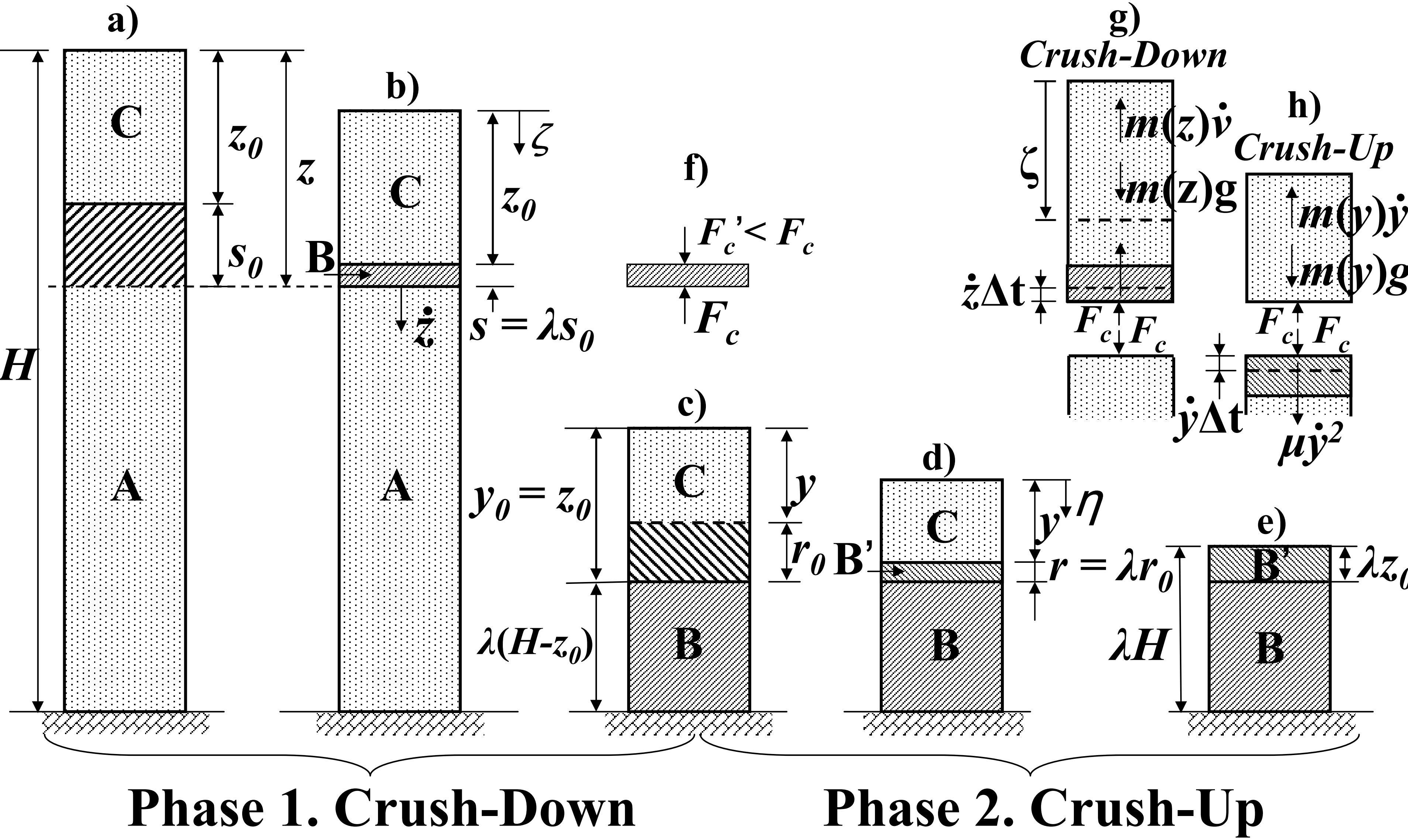
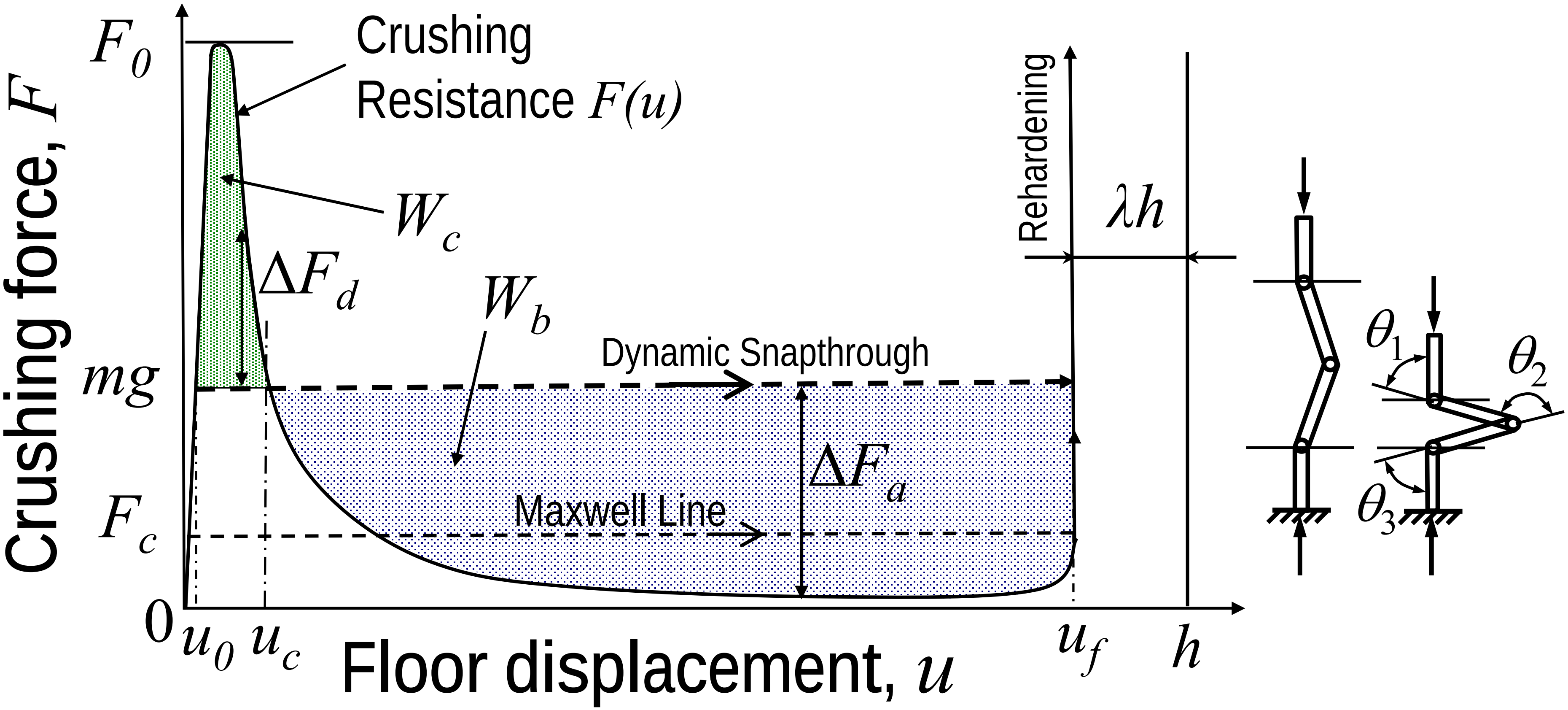


Fig. 1



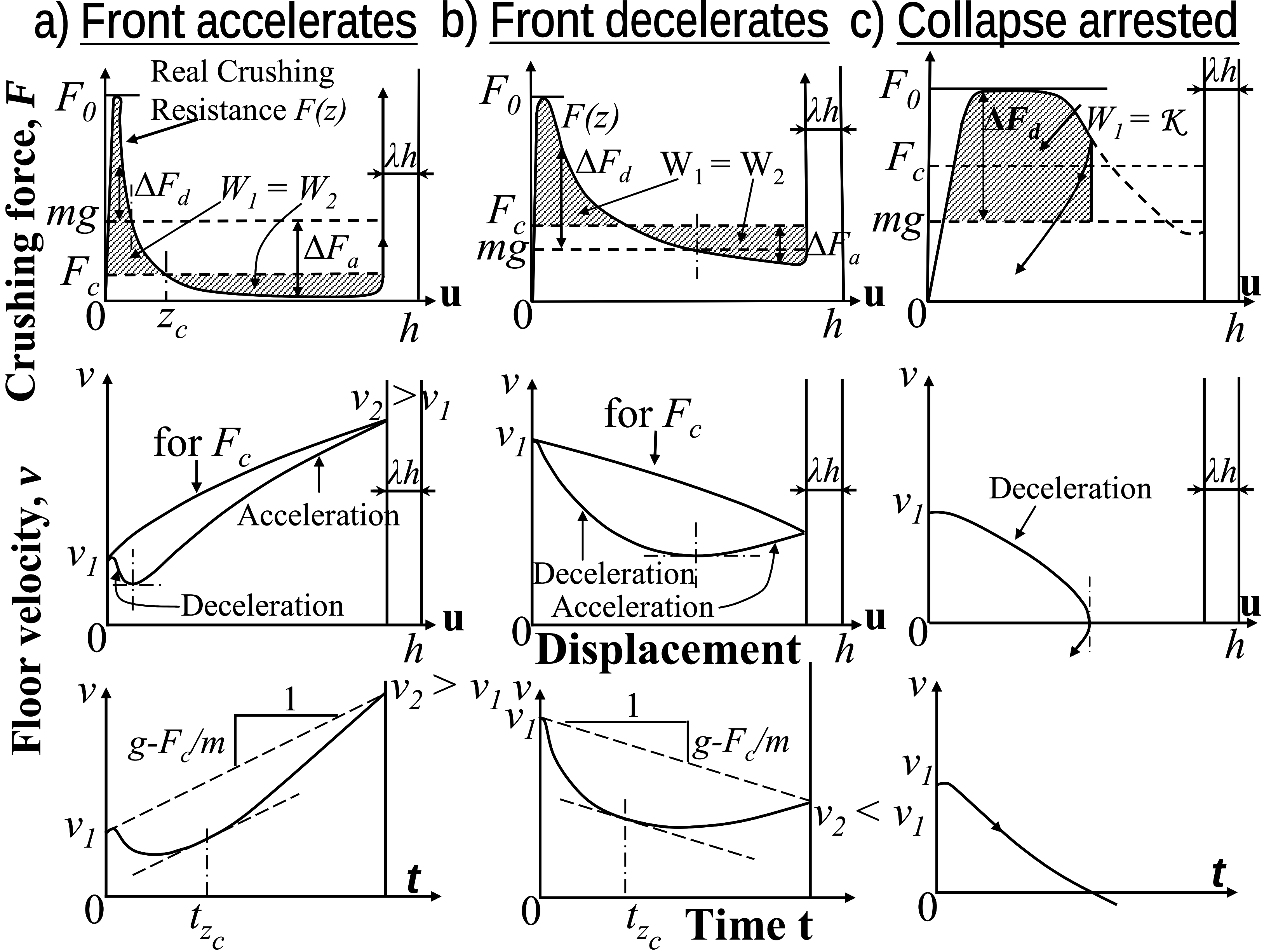
a)
b)
c)
d)
e)
f)
g)
Crush-Down
h)
Crush-Up
C
A
B
B'
H
z_0
z
s_0
ζ
$\dot{z}$
$s = \lambda s_0$
$F_c' < F_c$
F_c
$y_0 = z_0$
y
r_0
$\lambda(H-z_0)$
η
$r = \lambda r_0$
λz_0
λH
$m(z)\dot{v}$
$m(z)g$
$\dot{z}\Delta t$
$m(y)\dot{y}$
$m(y)g$
$\dot{y}\Delta t$
$\mu \dot{y}^2$
Phase 1. Crush-Down
Phase 2. Crush-Up

Fig. 2



F_0
Crushing
Resistance $F(u)$
W_c
ΔF_d
W_b
Crushing force, F
mg
Dynamic Snapthrough
ΔF_a
F_c
Maxwell Line
Rehardening
λh
θ_1
θ_2
θ_3
0
u_0
u_c
Floor displacement, u
u_f
h

Fig. 3



a) Front accelerates
b) Front decelerates
c) Collapse arrested
Crushing force, F
Floor velocity, v
Real Crushing Resistance F(z)
F(z)
W1 = W2
W1 = 𝒦
ΔFd
ΔFa
mg
Fc
F0
λh
zc
h
u
for Fc
Acceleration
Deceleration
v2 > v1
v2 < v1
g-Fc/m
Displacement
Time t
tzc

Fig. 4

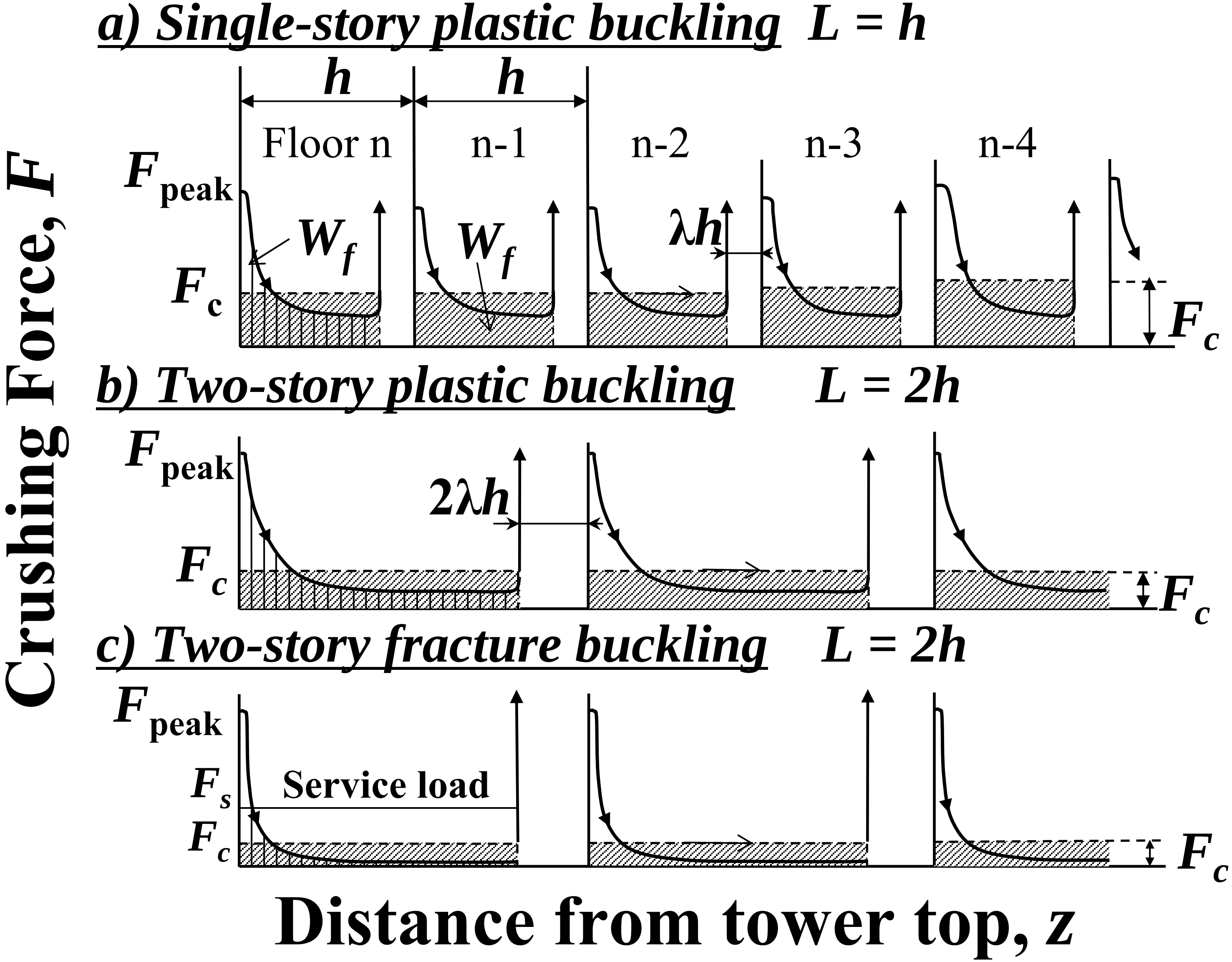
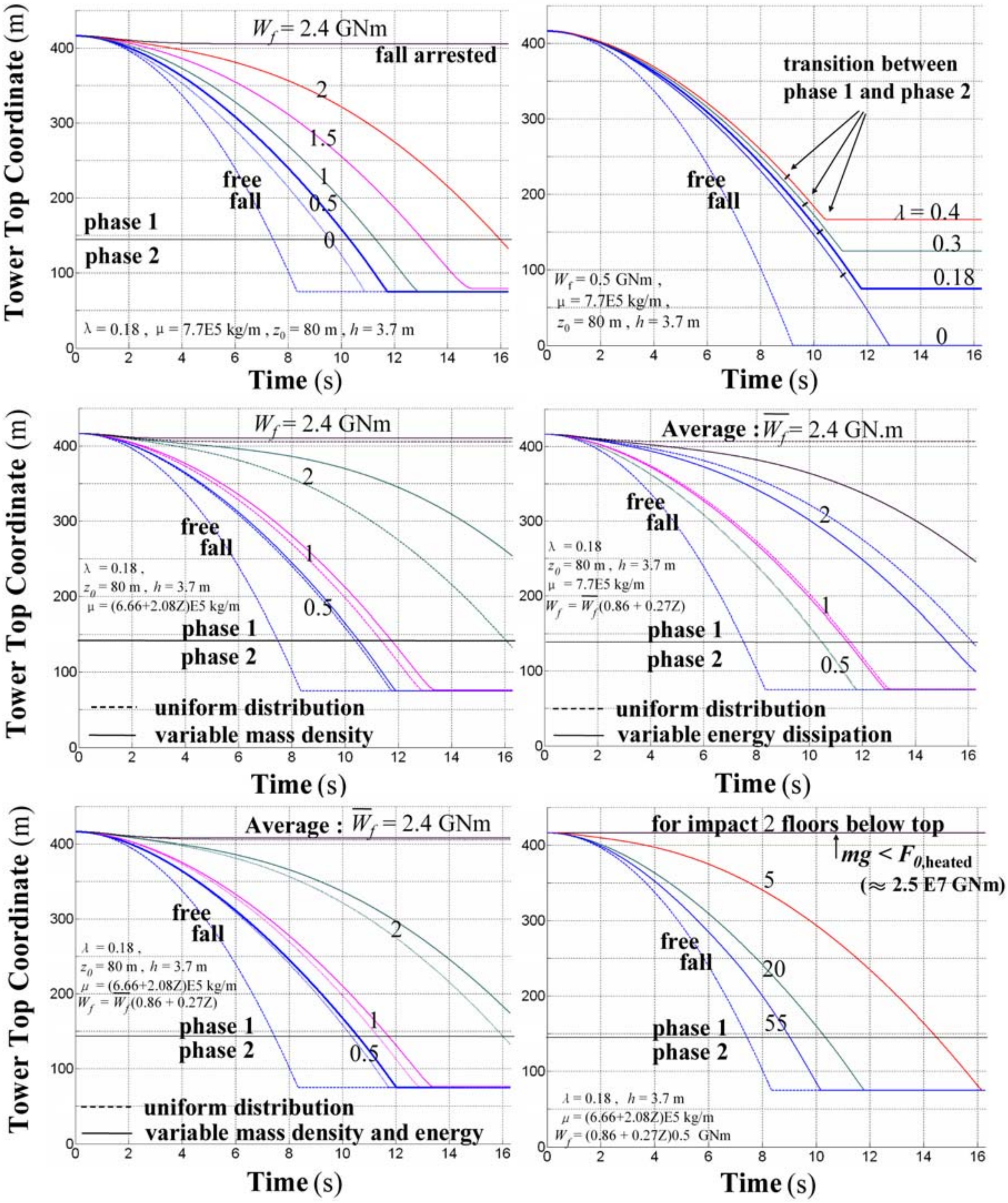

a) Single-story plastic buckling L = h
h
h
Floor n
n-1
n-2
n-3
n-4
F peak
W f
W f
λh
F c
F c
b) Two-story plastic buckling L = 2h
F peak
2λh
F c
F c
c) Two-story fracture buckling L = 2h
F peak
F s
Service load
F c
F c
Crushing Force, F
Distance from tower top, z

Fig. 5



Tower Top Coordinate (m)
Time (s)
W_f = 2.4 GNm
fall arrested
free fall
phase 1
phase 2
λ = 0.18 , μ = 7.7E5 kg/m , z_0 = 80 m , h = 3.7 m
transition between phase 1 and phase 2
λ = 0.4
W_f = 0.5 GNm , μ = 7.7E5 kg/m , z_0 = 80 m , h = 3.7 m
λ = 0.18 , z_0 = 80 m , h = 3.7 m μ = (6.66+2.08Z)E5 kg/m
uniform distribution
variable mass density
Average : W_f = 2.4 GN.m
λ = 0.18 z_0 = 80 m , h = 3.7 m μ = 7.7E5 kg/m W_f = W_f(0.86 + 0.27Z)
variable energy dissipation
Average : W_f = 2.4 GNm
λ = 0.18 , z_0 = 80 m , h = 3.7 m μ = (6.66+2.08Z)E5 kg/m W_f = W_f(0.86 + 0.27Z)
variable mass density and energy
for impact 2 floors below top
mg < F_0,heated
(≈ 2.5 E7 GNm)
λ = 0.18 , h = 3.7 m μ = (6.66+2.08Z)E5 kg/m W_f = (0.86 + 0.27Z)0.5 GNm

Fig. 6

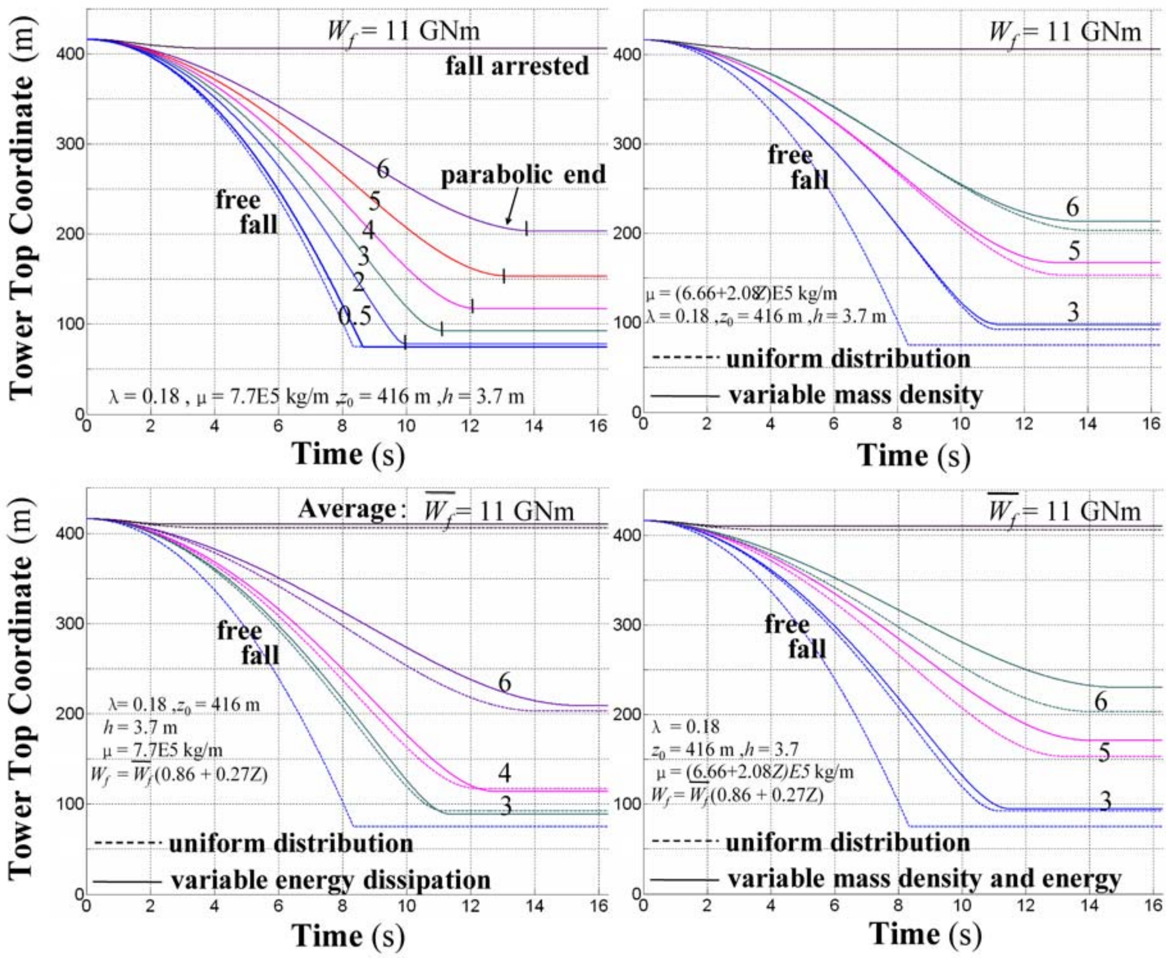

Tower Top Coordinate (m)
Time (s)
W_f = 11 GNm
fall arrested
parabolic end
free fall
6
5
4
3
2
0.5
λ = 0.18 , μ = 7.7E5 kg/m ,z_0 = 416 m ,h = 3.7 m
W_f = 11 GNm
free fall
μ = (6.66+2.08Z)E5 kg/m
λ = 0.18 ,z_0 = 416 m ,h = 3.7 m
uniform distribution
variable mass density
Average: W̄_f = 11 GNm
λ = 0.18 ,z_0 = 416 m
h = 3.7 m
μ = 7.7E5 kg/m
W_f = W̄_f(0.86 + 0.27Z)
uniform distribution
variable energy dissipation
W̄_f = 11 GNm
λ = 0.18
z_0 = 416 m ,h = 3.7
μ = (6.66+2.08Z)E5 kg/m
W_f = W̄_f(0.86 + 0.27Z)
uniform distribution
variable mass density and energy

Fig. 7